LAMPLIGHTER RANDOM WALKS

Wolfgang WOESS

Graz University of Technology, Austria

1. Introduction

X - an infinite graph.
(Graphs are locally finite and connected, neighbourhood $\sim$)

There is a lamp at each vertex of X.
States of the lamp (basic model) are $\{0, 1\} = \{$"off", "on"$\}$.
More generally: rooted graph $(L, 0)$, where root $0 \equiv$ "off".

Think of a lamplighter walking around X. At each step, he has the following options (or a combination thereof): at random

(a) move to an adjacent vertex, or
(b) change the state of the lamp to an adjacent state.

Configuration on X: function $\eta : X \to L$ with finite support

$$\mathit{supp}\,\eta = \{x \in X \mid \eta(x) \neq 0\}\,.$$

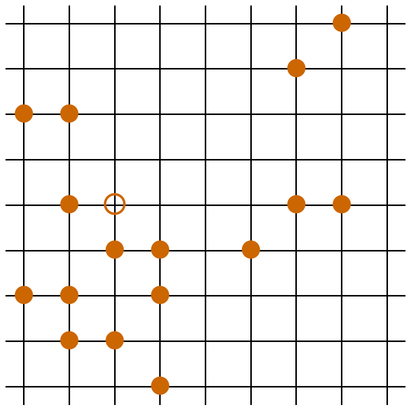
At each step, we have to observe the pair $\eta x := (\eta, x)\,,$ where η is the current configuration of the lamps and x is the current position of the lamplighter.

Let $\mathcal{C} = \mathcal{C}(X \to L) = \{\text{configurations}\}$. The state space of our random process is $L \wr X := \mathcal{C} \times X$ (wreath product).

An element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$:

Configuration η is $= 1$ at the •s

The lamplighter stands at the ○



The lamplighter graph has vertex set $L \wr X$ and a suitable neighbourhood relation; for example

Neighbourhood "walk or switch" $(\eta, x) \sim (\eta', x') :\Longleftrightarrow$

$$\begin{cases} x \sim x' \text{ in } X \text{ and } \eta = \eta' \quad \text{or} \\ x = x' \text{ in } X \text{ and } \begin{cases} \eta(x) \sim \eta'(x) \text{ in } L\,, \\ \eta(y) = \eta'(y)\ \forall\ y \neq x\,. \end{cases} \end{cases}$$

Neighbourhood "switch-walk-switch" $(\eta, x) \sim' (\eta', x') :\Longleftrightarrow$

$$x \sim x' \text{ in } X \text{ and } \begin{cases} \eta(x) \sim \eta'(x)\,,\ \eta(x') \sim \eta'(x') \text{ in } L \\ \eta(y) = \eta'(y)\ \forall\ y \neq x, x'\,. \end{cases}$$

Typical case: X and L are Cayley graphs of finitely generated groups Γ and K. Then $L \wr X$ is Cayley graph of the wreath product $K \wr \Gamma$.

Particular examples: $\mathbb{Z}_q \wr \mathbb{Z}^d$, and $\mathbb{Z}_2 \wr \mathbb{Z}$ (basic LL group).

2. Random walks

We consider random walks $Z_n = (Y_n, X_n)$ on $L \wr X$ whose transition probabilities are adapted to the wreath product (lamplighter) structure.

A. "Walk or switch" random walks

Let P_L and P_X be transition matrices on L and X , resp., and $0 < a < 1$. Define $P = P_a$ on $L \wr X$ by

$$p(\eta\, x, \eta'\, x') = \begin{cases} a\, p_X(x, x')\,, & \text{if } \eta' = \eta \text{ and } x' \neq x\,, \\ (1-a)\, p_L\Big(\eta(x), \eta'(x)\Big)\,, & \text{if } x' = x \text{ and } \eta' \neq_x \eta\,, \\ a\, p_X(x,x) + (1-a)\, p_L\Big(\eta(x), \eta(x)\Big)\,, & \text{if } x' = x \text{ and } \eta' = \eta\,, \\ 0\,, & \text{in all other cases.} \end{cases}$$

Interpretation. The lamplighter throws a coin. If "head" comes up (with probability a) then he makes a step in the base graph according to P_X and leaves all lamps unchanged. If "tails" comes up (with probability $1-a$) he does not move, but modifies the lamp at his position according to P_L.

Observations. • If P_L and P_X are irreducible (all states comunicate) then so is P.

• If L and X are regular and P_L and P_X are the respective simple random walks (SRWs) then SRW on $L \wr X$ is P_a with

$$a = \frac{deg_X}{deg_X + deg_L}.$$

• Write $Z_n = (Y_n, X_n)$ for the random walk corresponding to P. Then X_n is the random walk (Markov chain) on X with transition matrix

$$(1-a)\, I_X + a\, P_X .$$

B. “Switch - walk - switch” random walks

Let P_L and P_X be as above. Lift P_L and P_X to $L \wr X$ by

$$p_L(\eta\, x, \eta'\, x') = \begin{cases} p_L\big(\eta(x), \eta'(x)\big)\,, & \text{if } x' = x \text{ and } \eta' =_{X\setminus\{x\}} \eta\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$p_X(\eta\, x, \eta'\, x') = \begin{cases} p_X(x, x')\,, & \text{if } \eta' = \eta\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then we define Q (“switch - walk - switch”) on $L \wr X$ by

$$Q = P_L P_X P_L\,.$$

Observation. Write $Z_n = (Y_n, X_n)$ for the random walk corresponding to Q. Then X_n is the random walk (Markov chain) on X with transition matrix P_X.

Most important (key for asymptotic estimates):

Proposition [Varopoulos; Pittet and Saloff-Coste].
If L is finite and P_L is uniform on L, i.e., $p_L(\cdot,\cdot) = 1/|L|$, then the n-step return probabilities are

$$q^{(n)}(\eta\, x, \eta\, x) = \mathbf{E}_x\Big(|L|^{-R_n}\, \mathbf{1}_{\{X_n = x\}}\Big)\,.$$

Here,

$\mathbf{E}_x$ expectation on the trajectory space of (X_n) starting at x,

$R_n = |\{X_0, X_1, \dots, X_n\}|$ number of distinct points visited up to time n by the random walk on the base graph X.

Remark: random walk on groups

Let Γ be any group and μ a probability measure on Γ.

Random walk on Γ with law μ has transition probabilities

$$p(x,y) = \mu(x^{-1}y)\,.$$

If L and X are Cayley graphs of groups K and Γ, resp., and P_L and P_X come from random walks on these groups, then P (walk or switch) and Q (walk - switch - walk) are RWs on the wreath product group $K \wr \Gamma$.

3. Asymptotics for random walks on groups

Asymptotic type $\approx$

Let (a_n), (b_n) be two positive, monotone sequences. Write

$$a_n \preccurlyeq b_n :\Longleftrightarrow a_n \le C\, b_{cn} \quad (c, C > 0)\,, \quad \text{and}$$
$$a_n \approx b_n :\Longleftrightarrow a_n \preccurlyeq b_n \quad \text{and} \quad b_n \preccurlyeq a_n\,.$$

Theorem [Pittet and Saloff-Coste].

Asymptotic type is an invariant of finitely generated groups: for any two symmetric, irreducible, finite range random walks on such a group,

$$p_1^{(2n)}(x,x) \approx p_2^{(2n)}(x,x)$$

Theorem [Kesten; Varopoulos; Alexopoulos; Pittet].

For random walks on discrete subgroups of connected Lie groups, there are only three asymptotic types:

$n^{-d/2}$	if the group has polynomial growth with degree d,
$e^{-n^{1/3}}$	if the group is amenable with exponential growth,
e^{-n}	if the group is non-amenable.

Wreath products of groups $K \wr \Gamma$, i.e., lamplighter random walks on groups, provide a wealth of further examples of different asymptotic type.

Examples for symmetric, finite range RWs [Pittet and Saloff-Coste]

(a) On $F \wr \mathbb{Z}^d$, with finite group F

$$p^{(2n)}(x,x) \approx \exp\Big(-n^{d/(d+2)}\Big)$$

(b) On $\mathbb{Z} \wr \mathbb{Z}^d$,

$$p^{(2n)}(x,x) \approx \exp\Big(-n^{d/(d+2)}(\log n)^{2/(d+2)}\Big)$$

(c) If K is infinite with polynomial growth, then on $K \wr \mathbb{Z}$,

$$p^{(2n)}(x,x) \approx \exp\Big(-n^{1/3}(\log n)^{2/3}\Big)$$

(d) If K is polycyclic with exponential growth, then on $K \wr \mathbb{Z}$,

$$p^{(2n)}(x,x) \approx \exp\Big(-n^{1/2}\Big)$$

A more precise estimate for "switch - walk - switch" RW on the basic LL group $\mathbb{Z}_q \wr \mathbb{Z}$ [Revelle; Bartholdi and Woess]:

$$p^{(2n)}(x,x) \sim A \exp\Big(-B\, n^{1/3}\Big)\, n^{1/6}$$

4. Rate of escape

Let $d(\cdot,\cdot)$ denote the graph metric. For a finite range random walk (Z_n) with law μ on a group, it follows from Kingman's subadditive ergodic theorem [Guivarc'h; Derriennic] that there is a constant $\ell \geq 0$, the rate of escape, such that

$$\lim_{n\to\infty} \frac{d(Z_n, Z_0)}{n} \to \ell \quad \text{almost surely}$$

Recall: by the law of large numbers, $\ell = 0$ for finite range, symmetric RWs on $\mathbb{Z}^d$.

Theorem [Kaimanovich and Vershik]

For any finite range, symmetric, irreducible random walk on $F \wr \mathbb{Z}^d$ (F finite group) one has

$$\ell = 0 \quad \text{when} \quad d = 1,2, \quad \text{but} \quad \ell > 0 \quad \text{when} \quad d \geq 3.$$

Reason: when $d \le 2$, the random walk (X_n) of the lamplighter on the base graph $\mathbb{Z}^d$ is recurrent (returns almost surely to the starting point), while otherwise it is transient (visits each vertex only finitely often.)

More precisely: on groups $\ell > 0 \iff \exists$ non-constant bounded harmonic functions [Kaimanovich and Vershik; Varopoulos], see below.

This provided the first examples of symmetric RWs on amenable groups with positive rate of escape. There are deep & refined results on the rate of escape on LL groups by Erschler (Dyubina), e.g.:

Theorem [Erschler]

SRW on $\Gamma_1 = F \wr \mathbb{Z}^2$ (F finite) satisfies

$$\mathbf{E}\Big(d(Z_n, Z_0)\Big) \asymp n/\log n\,.$$

More generally, defining recursively $\Gamma_k = \Gamma_{k-1} \wr \mathbb{Z}^2$, on Γ_k

$$\mathbf{E}\Big(d(Z_n, Z_0)\Big) \asymp n/\log^{(k)} n\,,$$

where $\log^{(k)} n = \log\log^{(k-1)} n\,, \quad \log^{(1)} n = \log n\,.$

R. Lyons: Biased random walk RW_λ, modification of SRW on any graph X in terms of parameter $\lambda > 0$. If $x \in X \setminus o$ and $\mathit{deg}^-(x) = |\{v \sim x : d(v,o) = d(x,o) - 1\}|$ then for $y \sim x$

$$p_\lambda(x,y) = \begin{cases} \frac{\lambda}{\lambda\,\mathit{deg}^-(x) + (\mathit{deg}(x) - \mathit{deg}^-(x))}, & \text{if } d(y,o) = d(x,o) - 1 \\ \frac{1}{\lambda\,\mathit{deg}^-(x) + (\mathit{deg}(x) - \mathit{deg}^-(x))}, & \text{otherwise} \end{cases}$$

$\lambda = 1$ corresponds to SRW. In a Cayley graph of a group, if

$$\mathit{gr} = \lim_{n\to\infty} |\{x : d(x,o) \le n\}|^{1/n}$$

is the growth number, then RW_λ is transient for $\lambda < \mathit{gr}$ and recurrent for $\lambda > \mathit{gr}$; as λ increases, tends to become more homesick.

Theorem [Lyons, Pemantle and Peres]

On the "walk or switch" Cayley graph of the basic LL group $\mathbb{Z}_2 \wr \mathbb{Z}$, biased RW_λ satisfies

$$\lim_{n\to\infty} \frac{d(Z_n^\lambda, Z_0^\lambda)}{n} \to \begin{cases} 0\,, & \text{if } 0 < \lambda \le 1\,, \\ \ell(\lambda) > 0\,, & \text{if } 1 < \lambda < \mathit{gr} \end{cases} \qquad \text{almost surely.}$$

Back to “ordinary” LL random walks $Z_n = (Y_n, X_n)$ on $L \wr X$.

Important **observation**
(for studying rate of escape and boundary behaviour):

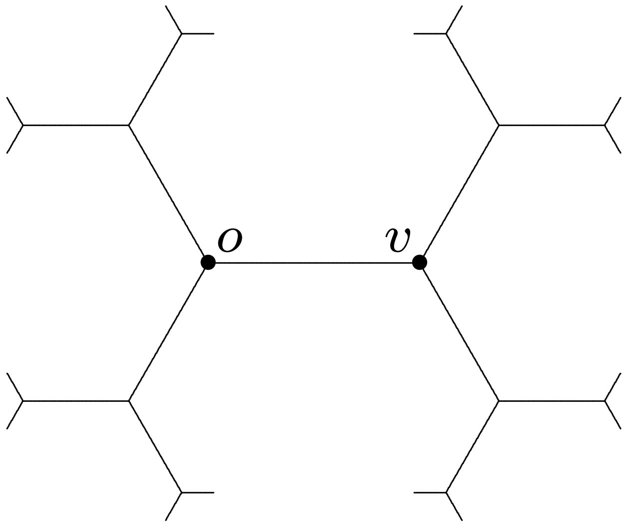
If the random walk (X_n) of the lamplighter on the base graph is transient there is a random limit configuration

$$Y_\infty = \lim_{n\to\infty} Y_n \in \mathcal{C}_\infty = \{\eta : X \to L\}$$

(not necessarily finitely supported).

$Y_\infty(x)$ is the definite state of the lamp at x. Indeed, x is visited by X_n only finitely often, and after the last visit, the state of the lamp at x remains unchanged.

Consider the "walk or switch" random walk $Z_n = (Y_n, X_n)$ with transition matrix P_a on $\mathbb{Z}_2 \wr \mathbb{T}$, where $\mathbb{T} = \mathbb{T}_q$ is the homogeneous tree with degree $q+1$.



$$p(\eta\, x, \eta'\, x') = \begin{cases} a/(q+1)\,, & \text{if } \eta' = \eta \text{ and } x' \sim x\,, \\ (1-a)\,, & \text{if } x' = x \text{ and } \eta' \neq_x \eta\,, \\ 0\,, & \text{in all other cases.} \end{cases}$$

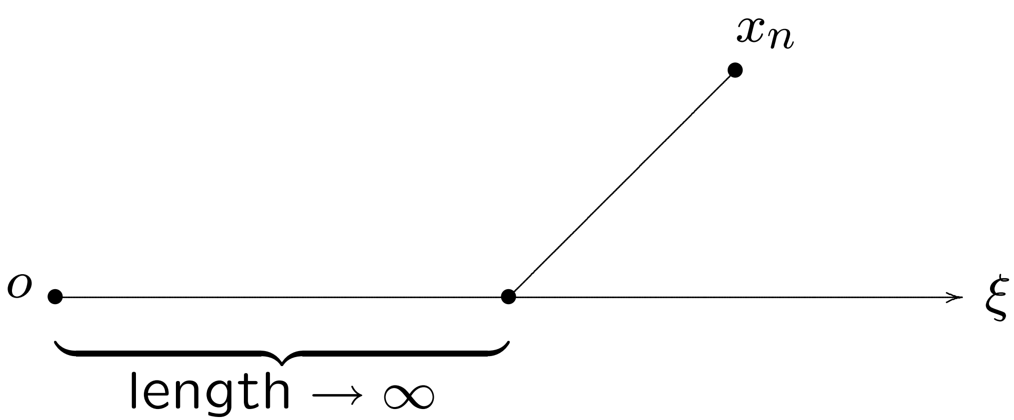
Well known & easy to compute: for the tree metric,

$$\ell_0 = \lim \frac{d(X_n, X_0)}{n} = a\frac{q-1}{q+1}\,.$$

X_n converges to a random end $X_\infty \in \partial\mathbb{T}$.

Ends (boundary points) $\xi \in \partial\mathbb{T}$ are represented by geodesic rays $\xi = [o = x_0, x_1, \ldots]$ starting from a root $o \in \mathbb{T}$.

Convergence of $x_n \in \mathbb{T}$ to $\xi \in \partial\mathbb{T}$:



Let $v \sim o$ be fixed. Let
$C_v = \{z \in \mathbb{T} \cup \partial\mathbb{T} : v$ lies on the geodesic from o to $z\}$. Define

$$\alpha = \mathbf{P}r[X_\infty \notin C_v\,,\ Y_\infty \neq 0 \text{ on } C_v]$$

(independent of choice of $v \sim o$).

Theorem [Gilch] In the "walk or switch" graph metric on $\mathbb{Z}_2 \wr \mathbb{T}_q$,

$$\ell = \lim \frac{d(Z_n, Z_0)}{n} = \ell_0 \left(1 + 2\alpha(q+1) + \frac{(1-a)(q+1)}{(1-a)(q+1) + (q-1)} \right)$$

The number α seems hard to compute explicitly, but there are good numerical estimates.

In particular $\ell > \ell_0$ strictly ("acceleration").
This strict inequality also holds for simple "switch – walk – switch" random walk on $\mathbb{Z}_2 \wr \mathbb{T}$.

In more recent work Gilch is studying this acceleration prenomenum for more general lamplighter RWs on groups and transitive graphs.

5. Bounded harmonic functions, Poisson boundary

Given X and the (irreducible) transition matrix P of a random walk on X, a harmonic function is a function $h : X \to \mathbb{R}$ such that

$$Ph = h\,, \quad \text{where} \quad Ph(x) = \sum_y h(y)$$

If the RW is recurrent then all positive (super)harmonic functions are constant.

Recall that for any random walk $p(x,y) = \mu(y-x)$ on $\mathbb{Z}^d$, all bounded harmonic functions are constant.

Theorem [Kaimanovich]
For a finite range (finite 1st moment suffices), symmetric, irreducible random walk on the group $\mathbb{Z}_2 \wr \mathbb{Z}^d$,

(a) all bounded harmonic functions are constant, if $d \in \{1,2\}$;

(b) there are non-constant bounded harmonic functions, if $d \geq 3$.

Outline of Proof (a) uses entropy criterion [Avez; Kaimanovich and Vershik; Derriennic].

(b) Let $Z_n = (Y_n, X_n)$ be the LL RW on $\mathbb{Z}_2 \wr \mathbb{Z}^d$ and X_n its projection on $\mathbb{Z}^d$. It is transient. Therefore (see above)

$$Y_n \to Y_\infty \in \mathcal{C}_\infty \quad \text{almost surely.}$$

Thus, if we define for $\eta\, x \in \mathbb{Z}_2 \wr \mathbb{Z}^d$ and $i = 0, 1$

$$\begin{aligned} h_i(\eta x) &= \Pr[\text{Lamp at origin is definitely in state } i \mid Z_0 = \eta\, x] \\ &= \Pr[Y_\infty(0) = i \mid Z_0 = \eta\, x] \end{aligned}$$

then $0 \le h_i$, $h_0 + h_1 \equiv 1$, and both functions are non-constant, bounded and harmonic.

The Poisson boundary associated with an irreducible random walk is a measure space Π with a family of mutually absolutely continuous probability measures $(\nu_x)_{x \in X}$, such that every bounded harmonic function h has a unique representation

$$h(x) = \int_{\Pi} \varphi \, d\nu_x \quad \text{with} \quad \varphi \in L^{\infty}(\Pi, \nu_{\cdot})$$

The Poisson boundary is trivial (each ν_x is the same point mass) iff all bounded harmonic functions are constant.

In most typical cases, one has some boundary at infinity,
that is, a set B and a metric (topology) on $X \cup B$, such that

$$\lim_{n\to\infty} Z_n = Z_\infty$$

exists almost surely in that topology, and ν_x is the associated limit distribution

$$\nu_x(B) = \mathbf{P}r[Z_\infty \in B \mid Z_0 = x]\,.$$

One wants to understand whether this is the finest model for distinguishing limit points at infinity. Now,

$$h = \int_B \varphi \, d\nu_x \quad \text{with} \quad \varphi \in L^\infty(B, \nu_\cdot)$$

always defines bounded harmonic functions, and we have the finest model (the Poisson boundary) if and only if every bounded harmonic function can be obtained in this way.

One typically wants to identify Π up to isomorphism of measure spaces.

Theorem [Kaimanovich]

For any finite range, irreducible random walk $Z_n = (Y_n, X_n)$ on $F \wr \mathbb{Z}^d$ (F finite group) for which X_n has non-zero drift $\mathbf{b} = \mathbf{E}(X_n - X_{n-1}) \neq \mathbf{0}$, the Poisson boundary is the space of all infinite configurations $\mathcal{C}_\infty$.

More precisely, let

$$\mathcal{C}_\mathbf{b} = \mathcal{C} \cup \left\{ \eta : \mathbb{Z}^d \to F \,\middle|\, \textit{supp}(\eta) = \{x_n : n \in \mathbb{N}\} \quad \text{with} \quad \frac{x_n}{|x_n|} \to \frac{\mathbf{b}}{|\mathbf{b}|} \right\}$$

Then

$$\frac{X_n}{|X_n|} \to \frac{\mathbf{b}}{|\mathbf{b}|} \quad \text{and} \quad Y_n \to Y_\infty \in \mathcal{C}_\infty(\mathbf{b}) \quad \text{almost surely.}$$

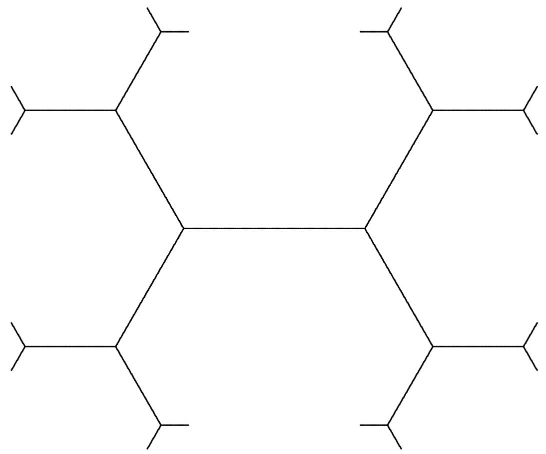
With the corresponding limit distributions, one gets Π.

When $\mathbf{b} = \mathbf{0}$ and $d \geq 3$, one also conjectures that $\Pi = \mathcal{C}_\infty$.

Let Γ be a group whose Cayley graph is the homogeneous tree $\mathbb{T} = \mathbb{T}_q$.

Consider "finite range" RWs $Z_n = (Y_n, X_n)$ on the wreath product group $F \wr \mathbb{T}$ (F a finite group).

Specific examples: simple "walk or switch" or "switch – walk – switch" RWs on $\mathbb{T}$.



X_n is a RW on the group Γ $\equiv$ on the tree $\mathbb{T}$. It converges to a random end:

$$\lim X_n = X_\infty \in \partial\mathbb{T}$$

Let

$$\mathcal{C}_\xi = \mathcal{C} \cup \mathcal{C} \cup \left\{ \eta : \mathbb{T} \to F \,\middle|\, \textsf{supp}(\eta) = \{x_n : n \in \mathbb{N}\} \quad \text{with} \quad x_n \to \xi \right\}$$

If $X_n \to \xi$ (for some trajectory) then $Y_\infty \in \mathcal{C}_\xi$.

Theorem [Karlsson and Woess]

$Z_n = (Y_n, X_n)$ converges to a limit random variable

$$Z_\infty \in \Pi := \left\{ (\eta, \xi) : \xi \in \partial\mathbb{T}\,,\ \eta \in \mathcal{C}_\xi \right\}.$$

Π together with the associated family of limit distributions is the Poisson boundary of the random walk.

Proof uses "strip criterion" of Kaimanovich.