# An invariance principle for random planar maps

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# Outline

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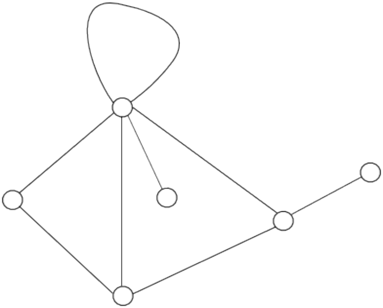
# Planar maps

## Definition

A planar map is an embedding of a connected graph in the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere. Loops and multiple edges are allowed



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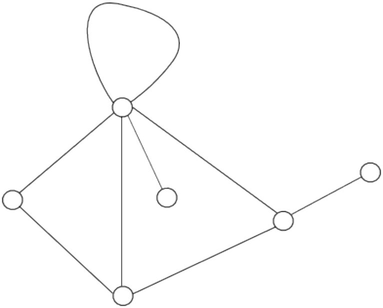

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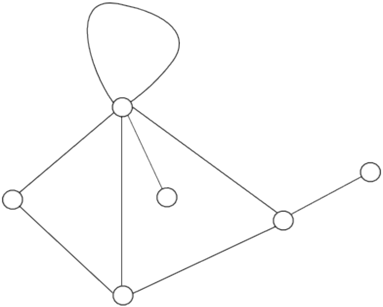

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# A brief History of the enumeration of maps

- Using generating functions and...
  - ...the quadratic method and Lagrange's inversion formula: enumeration of numerous families of maps by [Tutte, 1963]
  - ...fine analytic methods allow to obtain fine results on the structure (core size, connectivity) of large random maps [Banderier, Flajolet, Schaeffer, Soria 2001]
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# Motivations

- Partly motivated by physics, one can interpret a planar map as a particular discrete ‘geometrization’ of it.
- A simple way of doing that is to turn the map into a metric space, where vertices are endowed with the graph distances.
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# Random metrics

One is interested in the structure of (say) random triangulations $T_n$ with $n$ faces as $n \to \infty$. Two approaches:

- Take local limits: fix an origin in $T_n$, assign all edges a unit length, and let $n \to \infty$ looking from the origin. Yields a 'uniform' random discretization of the plane [Angel, Schramm 2003]
- Take scaling limits: assign a $o(1)$ length to each edge, so that one gets convergence as $n \to \infty$ to a random non-trivial object, the 'uniform spherical surface'. One should take this length of the order 'average distance between two vertices'.

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# Achievements

Local limits:

- Random uniform triangulations of the (half-)plane, growth properties, percolation: [Angel & Schramm 2003], [Angel 2003, 2005]. Methods are in the spirit of Tutte's enumeration schemes.
- Distributional convergence of the hull sizes in terms of discrete branching processes for triangulations and quadrangulations [Krikun 2005, 2006].
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- Another approach to these results is to define a conditioned version of the Brownian snake, and show invariance principles for positively-labelled Galton-Watson trees [Le Gall 2006, Le Gall & Weill 2006].
- Conjectural scaling limit for random quadrangulations, the Brownian map [Marckert & Mokkadem 2006].
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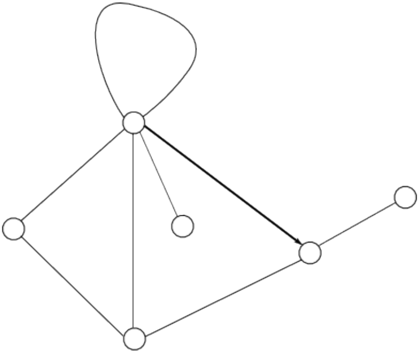
A planar map **m** is

- rooted if some oriented edge is distinguished as the base edge
- pointed if some vertex $r$ is distinguished as the base vertex



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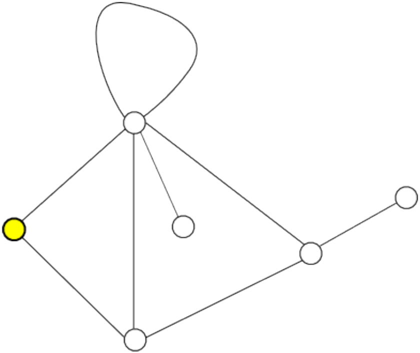

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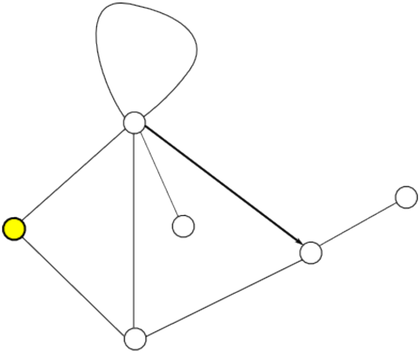

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# Boltzmann laws

- Fix a non-negative weight sequence $\mathbf{q} = (q_1, q_2, \ldots)$, and for $\mathbf{m} \in \mathcal{M}$, define its weight

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- If $Z_{\mathbf{q}} := W_{\mathbf{q}}(\mathcal{M})$ is finite, define a probability measure

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# The example of $\kappa$-angulations

- If $q_\kappa = q$ and $q_i = 0$ for $i \neq \kappa$, one gets $W_{\mathbf{q}}(\{\mathbf{m}\}) \neq 0$ iff all faces are $\kappa$-angles, and

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is finite whenever $0 \leq q \leq q_\kappa$.

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**Theorem (M. 2006)**

*On certain regularity assumptions on* $\mathbf{q}$*, the distribution of the rescaled radius*

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*converges weakly to* $C_{\mathbf{q}}\Delta$*, where* $C_{\mathbf{q}} > 0$ *is constant and* $\Delta$ *is the diameter of the Brownian snake.*

The constant $C_{\mathbf{q}}$ is in general hard to compute. In the case of $2\kappa$-angulations (easy),

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for triangulations (tedious)

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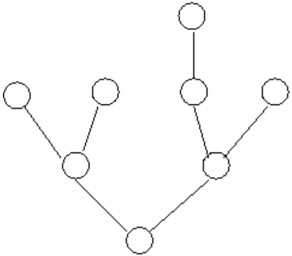
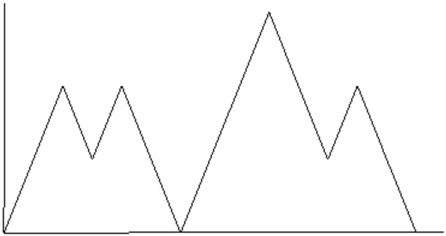
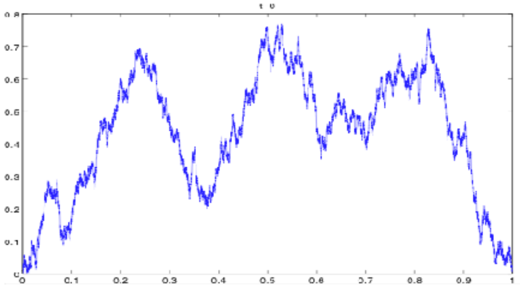
in general, is it true that

$$C_{\kappa} = \left( \frac{\kappa(\kappa - 2)}{9} \right)^{1/4} \quad ??$$

# The Brownian tree







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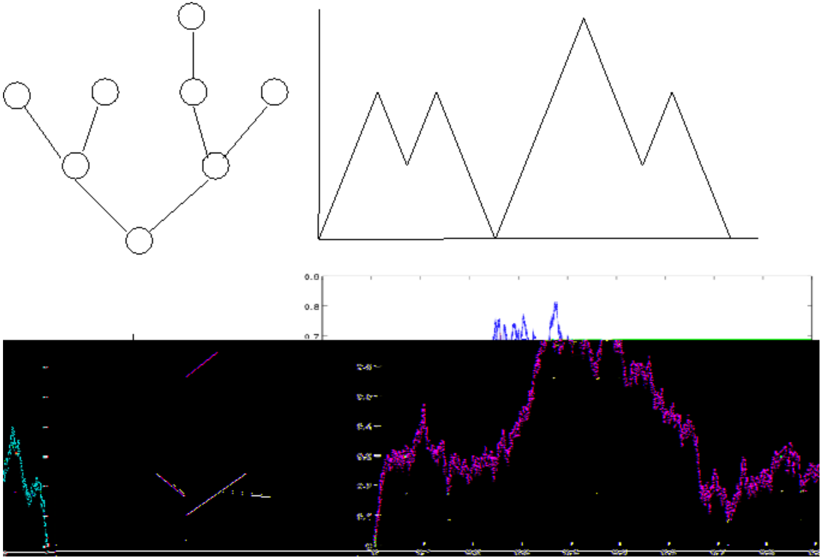

# The Brownian snake

- Assign each edge $e$ a random variable $Y_e$, independently over edges. Let $S_v$ be the sum of the $Y_e$ for $e$ belonging to the ancestral line of vertex $v$.
- This defines a random function on the vertices of the tree. If the $Y_e$ are i.i.d. centered with finite variance, then we sum about $n^{1/2}$ of them along each branch, so this converges to a Normal law when scaled by $n^{-1/4}$.
- Yields a random function on the continuum tree, which is an integrated white noise. This function $r$ is the Brownian snake. Its diameter is $\max r - \min r$.
- In general, the discrete snakes do converge in a strong sense to the continuous snake in a weak sense that does not entail convergence of diameter. Stronger results are proved in [Janson & Marckert 2004], under the (best) assumption $P(Y_e > t) = o(t^{-4})$.

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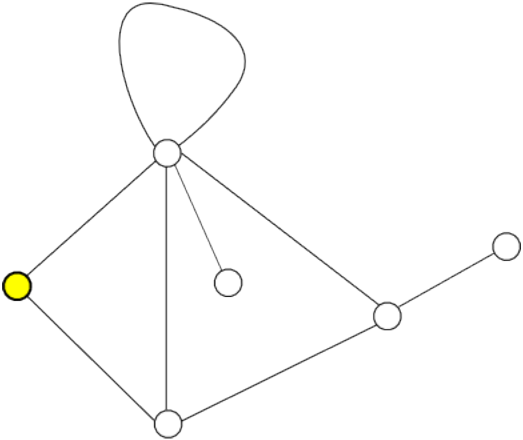
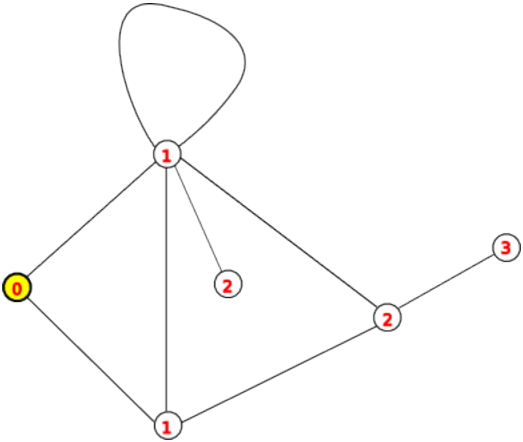
# The Bouttier-Di Francesco-Guitter bijection.

There is a family of bijection between planar maps and certain decorated and labelled trees (‘mobiles’). These generalise previous bijections by Schaeffer, who considers quadrangulations.

**1.** Start from the pointed map

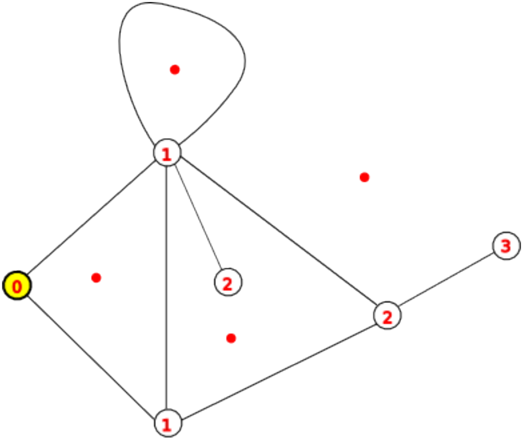


**2.** Label vertices by their distance to the base vertex



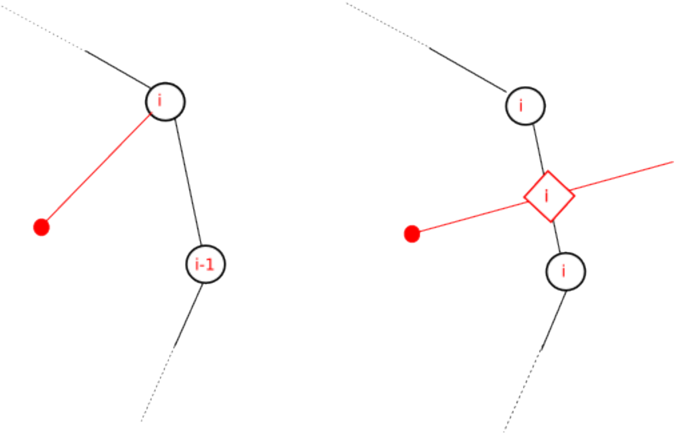

**3.** The dual vertices enter the picture

**2.** Label vertices by their distance to the base vertex

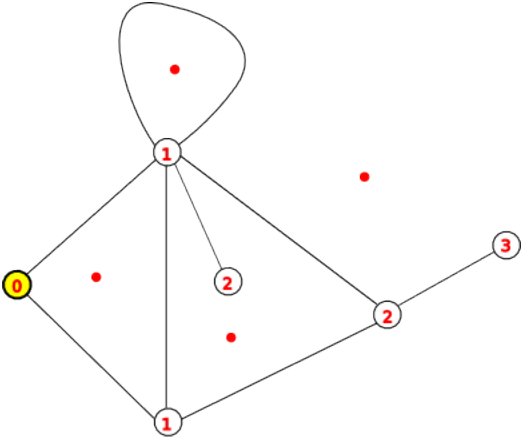


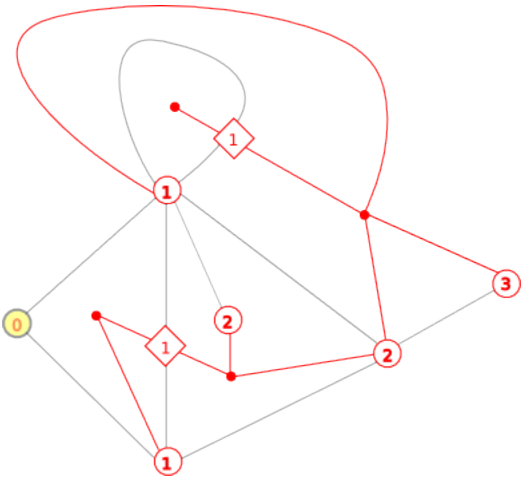

**3.** The dual vertices enter the picture

**4.** Turn around the red vertex inside each face, and add red edges according to the following two situations:



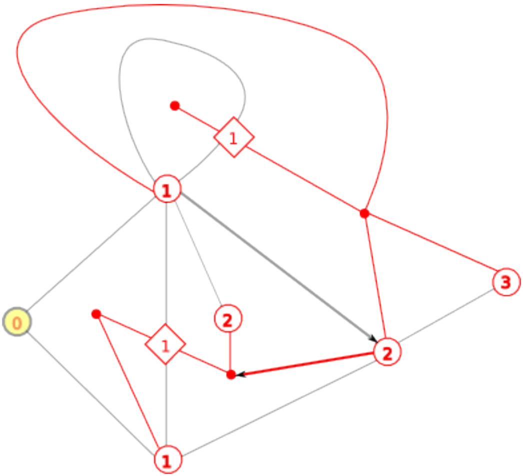
Here is what we obtain





1
1
1
0
2
1
2
3
1

**4.** Rooting the map allows to root the red tree



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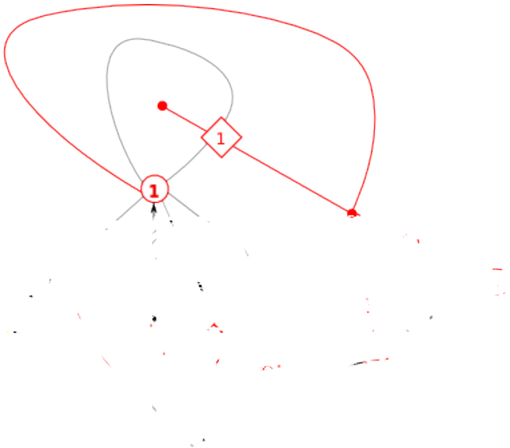
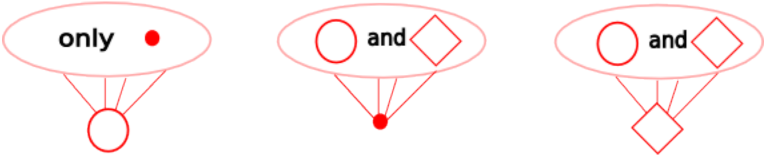


# The BDFG bijection

Up to minor modifications, one ends with

- a rooted tree with three kinds of vertices $\bigcirc, \bullet, \diamond$.

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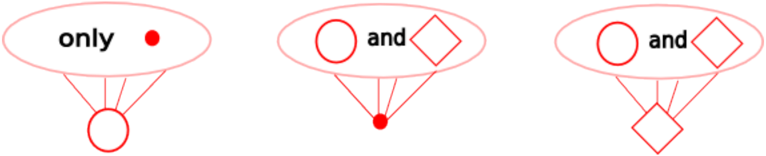


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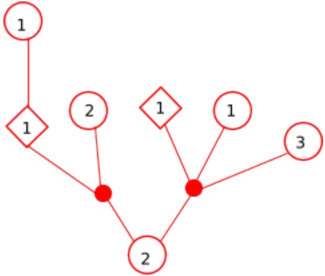

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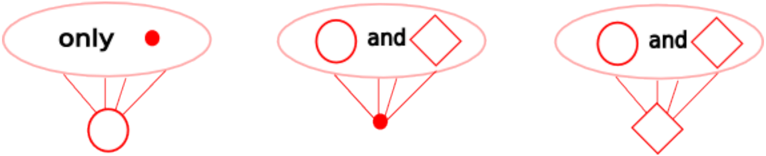
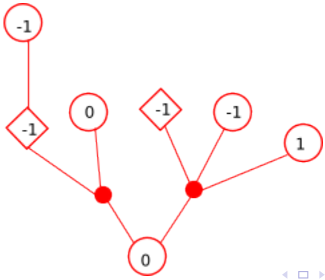
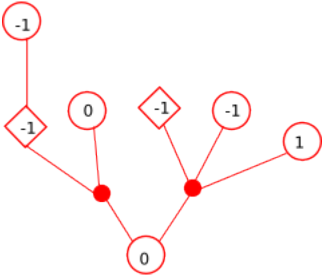



- positive labels on the vertices, with certain tricky labeling rules when going from a vertex to its sons. Subtract the label of the root to remove the positivity constraint on labels



- The trees thus obtained can further be simplified when the map is bipartite (only two alternating types remain) or a quadrangulation (only one type remains, the labeling constraints become easy).
- Note that the radius of the map (in general, all the information about distances to the base vertex) can be recovered fom the labels. Precisely, it equals the diameter of the support of the labeling function $+1$.



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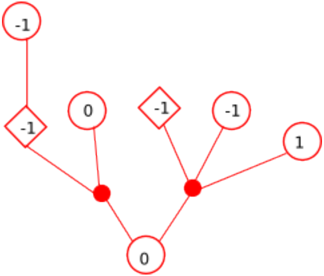

# Outline

# Multitype spatial Galton-Watson trees

- A $K$-type GW tree is a random tree whose vertices have $K$ different types, and such that each individual has a random number of offspring of the different types, independently over distinct individuals.
- These trees were introduced by Kolmogorov, and are well-understood under the assumption of irreducibility. Let

$$m_{ij} = \mathrm{ave}\#\text{offspring of type } j \text{ born from a type-} i \text{ individual.}$$

$M = (m_{ij})$ is the mean matrix, it is irreducible if for all $i, j$, $m_{ij}^{(n)} > 0$ for some $n$. Irreducible MGW trees 'behave like' usual Galton-Watson trees, the criticality condition is whether $\rho(M) \leq 1$ or $> 1$. See [Athreya & Ney 1972, Vatutin & Dyakonova 2001], [Kurtz, Lyons, Pemantle and Peres 1997], ...
- If the irreducibility fails, many behaviors can arise ([Janson 2005])

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## Fact

*If M is a Boltzmann-distributed map, then its image by the BDFG bijection is*

- *An irreducible Galton-Watson tree with three types of vertices: each type has its own offspring distribution, and*
- *a certain random labeling of the tree, so that the label differences along the edges are only locally dependent, and depend on the types of surrounding vertices.*

This entails the invariance principle for the radius of the map provided one is able to show the convergence of discrete spatial trees to the Brownian snake.

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The invariance principles of Janson & Marckert and Marckert & M. can be generalized to spatial irreducible multitype Galton-Watson trees, assuming [M., 2006]

1. Existence of small moments for the offspring distributions
2. Existence of a moment of order $8 + \varepsilon$ for the spatial displacements across the edges, that grows polynomially with the number of brothers of the edge.

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# Open questions and developments

1. Distribution of inter-distances of $n$ randomly sampled vertices (related to the uniqueness of the scaling limit).
2. Random metrics coupled with statistical physics models (percolation, Ising, RW, SAW...)
3. Radius when measuring distances from the origin of the root edge?
4. Imposing connectivity assumptions (2- or 3-connectedness)?
5. Better methods to compute the scaling constants?
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