

# Computing generating functions of ordered partitions with the transfer-matrix method

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An ordered partition of  $[n] := \{1, 2, \dots, n\}$  is a sequence of disjoint and nonempty subsets, called blocks, whose union is  $[n]$ . The aim of this paper is to compute some generating functions of ordered partitions by the transfer-matrix method. In particular, we prove several conjectures of Steingrímsson, which assert that the generating function of some statistics of ordered partitions give rise to a natural  $q$ -analogue of  $k!S(n, k)$ , where  $S(n, k)$  is the Stirling number of the second kind.

**Keywords:** Ordered partitions, Euler-Mahonian statistics,  $q$ -Stirling numbers of second kind, transfer-matrix method, determinants.

## Contents

<b>1</b>	<b>Introduction</b>	<b>193</b>
1.1	Background . . . . .	193
1.2	Definitions . . . . .	194
1.3	Main results . . . . .	196
<b>2</b>	<b>Ordered partitions and walks in digraphs</b>	<b>197</b>
2.1	Encoding ordered partitions by walks . . . . .	197
2.2	Generating functions of walks . . . . .	198
<b>3</b>	<b>Determinantal computations</b>	<b>200</b>
3.1	Proof of Theorem 2.6 . . . . .	200
3.2	Proof of Theorem 2.7 . . . . .	201

## 1 Introduction

### 1.1 Background

**Definition 1.1** A partition  $\pi_0$  of  $[n] = \{1, 2, \dots, n\}$  is a collection of disjoint and nonempty subsets, called blocks, whose union is  $[n]$ . By convention, we write  $\pi_0 = B_1 - B_2 - \dots - B_k$ , where the blocks  $B_i$  are arranged in increasing order of their minimal elements and within each block the elements are arranged in increasing order.

For instance,  $\pi_0 = 1 - 246 - 35 - 78$  is a partition of  $[8]$  with 4 blocks.

Let  $\mathcal{P}_n^k$  be the set of partitions of  $[n]$  with  $k$  blocks. In the present paper we study statistics on *ordered partitions*, that is, partitions where the blocks are ordered arbitrarily.

**Definition 1.2** An ordered partition  $\pi$  of  $[n]$  with  $k$  blocks is just a rearrangement of blocks of a partition in  $\mathcal{P}_n^k$ , that is there exist  $\pi_0 = B_1 - B_2 - \dots - B_k \in \mathcal{P}_n^k$  and  $\sigma$  a permutation of  $[k]$  such that  $\pi = B_{\sigma(1)} - B_{\sigma(2)} - \dots - B_{\sigma(k)}$ . We will say that  $\sigma$  is the permutation induced by  $\pi$  and set  $\sigma = \text{perm}(\pi)$ .

For instance,  $\pi = 35 - 246 - 1 - 78$  is an ordered partition of  $[8]$  with 4 blocks and we have  $\text{perm}(\pi) = 3214$ . Let  $\mathcal{OP}_n^k$  be the set of ordered partitions of  $[n]$  into  $k$  blocks. It is well known that the cardinality of  $\mathcal{OP}_n^k$  is the Stirling number of the second kind  $S(n, k)$ . It follows that  $k!S(n, k)$  counts the ordered partitions of  $[n]$  with  $k$  blocks.

Let  $\sigma = \sigma(1)\sigma(2) \dots \sigma(n)$  be a permutation of  $[n]$ , the integer  $i \in [n - 1]$  is called a *descent* of  $\sigma$  if  $\sigma(i) > \sigma(i + 1)$ . The *Eulerian number*  $A(n, k)$  counts permutations of  $[n]$  with  $k$  descents. There is a basic identity relating the  $S(n, k)$ 's and the  $A(n, k)$ 's, namely

$$k!S(n, k) = \sum_{m=1}^k \binom{n-m}{n-k} \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle, \tag{1}$$

which is easily proved combinatorially (for instance, see [10]).

In the present paper we will study some statistics originated from a  $q$ -analogue of (1) by means of the  $q$ -Eulerian numbers and  $q$ -Stirling numbers of the second kind, which were introduced by Carlitz [1, 2]. We need more definitions and notations.

Define the  $p, q$ -integer  $[n]_{p,q} = \frac{p^n - q^n}{p - q}$ , the  $p, q$ -factorial  $[n]_{p,q}! = [1]_{p,q}[2]_{p,q} \dots [n]_{p,q}$  and the  $p, q$ -binomial coefficients

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} \quad n \geq k \geq 0.$$

If  $p = 1$ , we shall write  $[n]_q, [n]_q!$  and  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_q$  for  $[n]_{1,q}, [n]_{1,q}!$  and  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_{1,q}$  respectively.

The  $q$ -Eulerian numbers  $A_q(n, k)$  ( $n \geq k \geq 0$ ) are defined by

$$A_q(n, k) = q^k [n-k]_q \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_q + [k+1]_q \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_q.$$

Let  $\sigma$  be a permutation of  $[n]$ , define the statistic  $\text{maj } \sigma = \sum_i i$  where the summation is over all descents of  $\sigma$ . Then

$$A_q(n, k) = \sum_{\sigma} q^{\text{maj } \sigma},$$

where the summation is over all permutations of  $[n]$  with  $k$  descents.

The  $q$ -Stirling numbers  $S_q(n, k)$  of the second kind are defined by:

$$S_q(n, k) = q^{k-1} S_q(n-1, k-1) + [k]_q S_q(n-1, k) \quad (n \geq k \geq 0), \tag{2}$$

where  $S_q(n, k) = \delta_{nk}$  if  $n = 0$  or  $k = 0$ .

A  $q$ -analogue of (1) has been found in [15] as follows:

$$[k]_q! S_q(n, k) = \sum_{m=1}^k q^{k(k-m)} \left[ \begin{matrix} n-m \\ n-k \end{matrix} \right]_q \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle_q. \tag{3}$$

There has been a considerable amount of recent interest in properties and combinatorial interpretations of the  $q$ -Stirling numbers and related numbers (see e.g. [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13]). In the aim to give a combinatorial proof of (3), Steingrímsson [10] introduced the following definition.

**Definition 1.3** A statistic *Stat* on  $\mathcal{OP}_n^k$  is called Euler-Mahonian if its generating function is equal to  $[k]_q! S_q(n, k)$ , i.e.,

$$\sum_{\pi \in \mathcal{OP}_n^k} q^{\text{Stat } \pi} = [k]_q! S_q(n, k).$$

Steingrímsson [10] has found a few of Euler-Mahonian statistics and has introduced new statistics on ordered partitions. Moreover, he has conjectured that all these new statistics are Euler-Mahonian. Wachs [11] has also obtained some *Euler-Mahonian* statistics on ordered partitions.

### 1.2 Definitions

Given a partition  $\pi$  in  $\mathcal{OP}_n^k$ , the elements of  $[n]$  are divided into four classes:

- *singletons*: elements of the singleton blocks;
- *openers*: smallest elements of the non singleton blocks;
- *closers*: largest elements of the non singleton blocks;
- *transients*: all other elements, i.e., non extremal elements of non singleton blocks.

The sets of openers, closers, singletons and transients of  $\pi$  will be denoted by  $\mathcal{O}(\pi)$ ,  $\mathcal{C}(\pi)$ ,  $\mathcal{S}(\pi)$  and  $\mathcal{T}(\pi)$ , respectively. The 4-tuple  $(\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$  is called the *type* of  $\pi$ . For instance, if  $\pi = 3\ 5 - 2\ 4\ 6 - 1 - 7\ 8$ , then

$$\mathcal{O}(\pi) = \{2, 3, 7\}, \quad \mathcal{C}(\pi) = \{5, 6, 8\}, \quad \mathcal{S}(\pi) = \{1\} \quad \text{and} \quad \mathcal{T}(\pi) = \{4\}.$$

**Definition 1.4** Let  $\sigma$  be a permutation of  $[n]$ ; the pair  $(i, j)$  is an inversion if  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . Let  $\text{inv } \sigma$  be the number of inversions in  $\sigma$ . We also define the number of inversions of an ordered partition  $\pi$ ,  $\text{inv } \pi$ , by  $\text{inv } \pi = \text{inv}(\text{perm}(\pi))$ . For a partition  $\pi$  in  $\mathcal{OP}_n^k$ , we also set  $\text{cinv } \pi = \binom{k}{2} - \text{inv } \pi$ .

Let  $\pi = B_1 - B_2 - \dots - B_k$  be a partition in  $\mathcal{OP}_n^k$ . We define a *partial order* on blocks  $B_i$ 's as follows :  $B_i > B_j$  if all the letters of  $B_i$  are greater than those of  $B_j$ ; in other words, if the opener of  $B_i$  is greater than the closer of  $B_j$ .

**Definition 1.5** A block inversion in  $\pi$  is a pair  $(i, j)$  such that  $i < j$  and  $B_i > B_j$ . We denote by  $\text{bInv } \pi$  the number of block inversions in  $\pi$ . We also set  $\text{cbInv} = \binom{k}{2} - \text{bInv}$ .

Let  $w_i$  be the index of the block (counting from the left) containing  $i$ , namely the integer  $j$  such that  $i \in B_j$ . Following Steingrímsson [10], we define for  $1 \leq i \leq n$  ten coordinate statistics on  $\pi \in \mathcal{OP}_n^k$ :

$$\begin{aligned} \text{ros}_i(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid j < i, w_j > w_i\}, \\ \text{rob}_i(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid j > i, w_j > w_i\}, \\ \text{rcs}_i(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid j < i, w_j > w_i\}, \\ \text{rcb}_i(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid j > i, w_j > w_i\}, \\ \text{los}_i(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid j < i, w_j < w_i\}, \\ \text{lob}_i(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \mid j > i, w_j < w_i\}, \\ \text{lcs}_i(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid j < i, w_j < w_i\}, \\ \text{lcb}_i(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \mid j > i, w_j < w_i\}, \end{aligned}$$

where  $(\mathcal{O} \cup \mathcal{S})(\pi) = \mathcal{O}(\pi) \cup \mathcal{S}(\pi)$ , and let  $\text{rsb}_i(\pi)$  (resp.  $\text{lsb}_i(\pi)$ ) be the number of blocks in  $\pi$  to the right (resp. left) of the block containing  $i$  such that the opener of  $B$  is smaller than  $i$  and the closer of  $B$  is greater than  $i$ . Then define  $\text{ros}, \text{rob}, \text{rcs}, \text{rcb}, \text{lob}, \text{los}, \text{lcs}, \text{lcb}, \text{lsb}$  and  $\text{rsb}$  as the sum of their coordinate statistics, e.g.

$$\text{ros} = \sum_i \text{ros}_i.$$

**Remark 1.6** Note that  $\text{ros}$  is the abbreviation of "right, opener, smaller", while  $\text{lsb}$  is the abbreviation of "left, closer, bigger", etc.

More generally, if  $\text{stat}$  is one of the above ten statistics, we define  $\text{stat}(\mathcal{O} \cup \mathcal{S})$  by

$$\text{stat}(\mathcal{O} \cup \mathcal{S})(\pi) = \sum_{i \in (\mathcal{O} \cup \mathcal{S})(\pi)} \text{stat}_i(\pi).$$

In the same way, we define  $\text{stat}(\mathcal{T} \cup \mathcal{C})$ ,  $\text{stat}(\mathcal{C} \cup \mathcal{S})$ , etc. The following results illustrate the above notions.

**Proposition 1.7** The following functional identities hold on  $\mathcal{OP}_n^k$ :

$$\text{bInv} = \text{rcs}(\mathcal{O} \cup \mathcal{S}), \quad \text{inv} = \text{ros}(\mathcal{O} \cup \mathcal{S}) \quad \text{and} \quad \text{cinv} = \text{los}(\mathcal{O} \cup \mathcal{S}). \tag{4}$$

For instance, we give the values of the coordinate statistics computed on the partition  $\pi = 6\ 8 - 5 - 1\ 4\ 7 - 3\ 9 - 2$ :

$$\begin{aligned} \pi &= 6\ 8 - 5 - 1\ 4\ 7 - 3\ 9 - 2 \\ \text{los}_i &: 0\ 0 - 0 - 0\ 0\ 2 - 1\ 3 - 1 \\ \text{ros}_i &: 4\ 4 - 3 - 0\ 2\ 2 - 1\ 1 - 0 \\ \text{lob}_i &: 0\ 0 - 1 - 2\ 2\ 0 - 2\ 0 - 3 \\ \text{rob}_i &: 0\ 0 - 0 - 2\ 0\ 0 - 0\ 0 - 0 \\ \text{lcs}_i &: 0\ 0 - 0 - 0\ 0\ 1 - 0\ 3 - 0 \\ \text{rcs}_i &: 2\ 3 - 1 - 0\ 1\ 1 - 1\ 1 - 0 \\ \text{lcb}_i &: 0\ 0 - 1 - 2\ 2\ 1 - 3\ 0 - 4 \\ \text{rcb}_i &: 2\ 1 - 2 - 2\ 1\ 1 - 0\ 0 - 0 \\ \text{lsb}_i &: 0\ 0 - 0 - 0\ 0\ 1 - 1\ 0 - 1 \\ \text{rsb}_i &: 2\ 1 - 2 - 0\ 1\ 1 - 0\ 0 - 0 \end{aligned}$$

Since  $\{6, 8\} > \{5\}$ ,  $\{6, 8\} > \{2\}$ ,  $\{5\} > \{2\}$  and  $\{3, 9\} > \{2\}$ , we have  $\text{bInv } \pi = 4$  and  $\text{cbInv } \pi = \binom{5}{2} - 4 = 6$ . Moreover,  $\text{perm}(\pi) = 54132$ , thus  $\text{inv}(\pi) = 8$  and  $\text{cinv}(\pi) = \binom{5}{2} - 8 = 2$ .

Inspired by the statistic  $\text{mak}$  on the permutations due to Foata & Zeilberger [3], Steingrímsson introduced its analogous on  $\mathcal{OP}_n^k$  as follows:

$$\begin{aligned} \text{mak} &= \text{ros} + \text{lcs}, \\ \text{lmak} &= n(k-1) - [\text{los} + \text{rcs}], \\ \text{mak}' &= \text{lob} + \text{rcb}, \\ \text{lmak}' &= n(k-1) - [\text{lcb} + \text{rob}]. \end{aligned}$$

The following proposition permits to reduce the conjecture 12 in [10] almost by half. This result was first proved in [4].

**Proposition 1.8** *On  $\mathcal{OP}_n^k$  the following functional identities hold:*

$$\text{mak} = \text{lmak}' \quad \text{and} \quad \text{mak}' = \text{lmak}.$$

Define also the statistic  $\text{cinvLSB}$  on  $\mathcal{OP}^k$  by

$$\text{cinvLSB} := \text{lsb} + \text{cbInv} + \binom{k}{2}. \tag{5}$$

### 1.3 Main results

Consider the following two generating functions of ordered partitions with  $k \geq 0$  blocks:

$$\phi_k(a; x, y, t, u) := \sum_{\pi \in \mathcal{OP}^k} x^{(\text{mak} + \text{bInv})\pi} y^{\text{cinvLSB } \pi} t^{\text{inv } \pi} u^{\text{cinv } \pi} a^{|\pi|}, \tag{6}$$

$$\varphi_k(a; z, t, u) := \sum_{\pi \in \mathcal{OP}^k} z^{(\text{lmak} + \text{bInv})\pi} t^{\text{inv } \pi} u^{\text{cinv } \pi} a^{|\pi|}, \tag{7}$$

where  $|\pi| = n$  if  $\pi$  is a partition of  $[n]$ . The following Theorem is the main result of this paper.

**Theorem 1.9** *We have*

$$\phi_k(a; x, y, t, u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{8}$$

$$\varphi_k(a; z, t, u) = \frac{a^k z^{\binom{k}{2}} [k]_{tz,u}!}{\prod_{i=1}^k (1 - a[i]_z)}. \tag{9}$$

The above theorem infers results on Euler-Mahonian statistics on ordered partitions. Indeed, it follows directly from (2) that

$$\sum_{n \geq k} [k]_q! S_q(n, k) a^n = \frac{a^k q^{\binom{k}{2}} [k]_q!}{\prod_{i=1}^k (1 - a[i]_q)}. \tag{10}$$

Then, by appropriate specializations in (6) and (7), we obtain the following result conjectured by Steingrímsson [10, conj.12 and 13].

**Theorem 1.10** *The following statistics are Euler-Mahonian on  $\mathcal{OP}_n^k$ :*

$$\text{mak} + \text{bInv}, \quad \text{lmak} + \text{bInv}, \quad \text{cinvLSB}.$$

*In other words, the generating functions of the above statistics over  $\mathcal{OP}_n^k$  are all equal to  $[k]_q! S_q(n, k)$ .*

Others specializations give rise to new Euler-Mahonian statistics.

**Theorem 1.11** *The following statistics are Euler-Mahonian on  $\mathcal{OP}_n^k$ :*

$$\text{mak} + \text{bInv} - (\text{inv} - \text{cinv}), \quad \text{lmak} + \text{bInv} + (\text{inv} - \text{cinv}), \quad \text{cinvLSB} + (\text{inv} - \text{cinv}).$$

Theorem 1.9 provides also an alternative proof of the following combinatorial interpretations for  $q$ -Stirling numbers of the second kind, where the first two interpretations were proved by Ksavrelof and Zeng [4] and the third interpretation was first proved by Stanton (see [12]).

**Corollary 1.12**

$$S_q(n, k) = \sum_{\pi \in \mathcal{P}_n^k} q^{\text{mak} \pi} = \sum_{\pi \in \mathcal{P}_n^k} q^{\text{lmak} \pi} = \sum_{\pi \in \mathcal{P}_n^k} q^{\text{lsb} \pi + \binom{k}{2}}.$$

Indeed, an (unordered) partition can be identified with an ordered partition without inversion, i.e.,

$$\mathcal{P}_n^k = \{\pi \in \mathcal{OP}_n^k \mid \text{inv} \pi = 0\}.$$

Since the statistic  $\text{bInv}$  vanishes on (unordered) partitions and the identity  $\text{cinvLSB} = \text{lsb} + 2\binom{k}{2}$  holds on  $\mathcal{P}^k$ , it is then trivial to obtain Corollary 1.12.

There are three main ingredients in the proof of Theorem 1.9: a bijection which maps ordered partitions to walks on some digraphs, the transfer-matrix method which permits to compute the generating function of walks using determinants, and the evaluation of determinants.

## 2 Ordered partitions and walks in digraphs

### 2.1 Encoding ordered partitions by walks

Let  $\pi = B_1 - B_2 - \dots - B_k$  in  $\mathcal{OP}_n^k$  and  $i$  an integer in  $[n]$ . The restriction  $B_j(\leq i) := B_j \cap [i]$  of the block  $B_j$  is said to be *active* if  $B_j \not\subseteq [i]$  and  $B_j \cap [i] \neq \emptyset$ , *complete* if  $B_j \subseteq [i]$ . We define the restriction of  $\pi$  on  $[i]$ , called the  $i$ -th *trace* of  $\pi$ , by

$$T_i(\pi) = B_1(\leq i) - B_2(\leq i) - \dots - B_k(\leq i),$$

with the empty restrictions being omitted. The sequence  $(T_i(\pi))_{1 \leq i \leq n}$  is called the *trace* of the partition  $\pi$ . We denote by  $\text{act}_i \pi$  and  $\text{com}_i \pi$  the numbers of active blocks and complete blocks, respectively, in  $T_i(\pi)$  and set

$$\omega_i(\pi) = (\text{com}_i \pi, \text{act}_i \pi) \quad \text{for } 1 \leq i \leq n,$$

with  $\omega_0(\pi) = (0, 0)$ . The sequence  $(\omega_i(\pi))_{0 \leq i \leq n}$  is called the *walk* of the partition  $\pi$ .

For instance, if  $\pi = \{6\} - \{3, 5, 7\} - \{1, 4, 10\} - \{9\} - \{2, 8\}$ , then  $T_6(\pi) = \{6\} - \{3, 5, \dots\} - \{1, 4, \dots\} - \{2, \dots\}$ , where each active block ends with an ellipsis, and we get  $\omega_6(\pi) = (1, 3)$ .

**Definition 2.1** *Let  $D = (V, E)$  be the digraph on  $V = \mathbb{N}^2$  with edges set  $E$  defined by*

$$E = \{(u, v) \in V^2 \mid u = v = (x, y) \text{ with } y > 0 \text{ or } u - v = (0, 1), (1, 0), (1, -1)\}.$$

*Clearly there are four types of edges. An edge  $(u, v)$  of  $D$  is called:*

- North if  $v = u + (0, 1)$ ;
- East if  $v = u + (1, 0)$ ;
- South-East if  $v = u + (1, -1)$ ;
- Null if  $v = u$ .

For any integer  $k \geq 0$ , let  $V_k = \{(i, j) \in V \mid i + j \leq k\}$  and  $D_k$  be the restriction of the digraph  $D$  on  $V_k$ . An illustration of  $D_k$  is given in Figure 1.

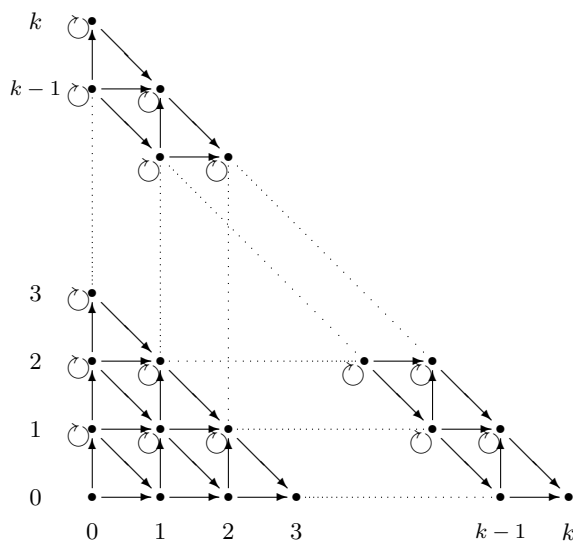


Fig. 1: The digraph  $D_k$

**Definition 2.2** A walk of depth  $k$  and length  $n$  in  $D$  is a sequence  $\omega = (v_0, v_1, \dots, v_n)$  of points in  $V$  such that  $v_0 = (0, 0)$ ,  $v_n = (k, 0)$  and  $(v_i, v_{i+1})$  is an edge of  $D$  for  $i = 0, \dots, n - 1$ . Moreover, the abscissa and height of the step  $(v_i, v_{i+1})$  are the abscissa and ordinate of  $v_i$  respectively. Let  $\Omega_n^k$  be the set of walks of depth  $k$  and length  $n$  and  $\Omega^k$  be the set of walks of depth  $k$  in  $D$ .

The following result characterizes the walks of ordered partitions.

**Proposition 2.3** For  $n \geq k \geq 0$ , the walks of the partitions in  $\mathcal{OP}_n^k$  are exactly the walks in  $\Omega_n^k$ .

We need a refinement of the notion of walk.

**Definition 2.4** A walk diagram of depth  $k$  and length  $n$  is a pair  $(\omega, \xi)$ , where  $\omega = (v_0, v_1, \dots, v_n)$  is a walk in  $\Omega_n^k$  and  $\xi = (\xi_i)_{1 \leq i \leq n}$  is a sequence of integers such that  $1 \leq \xi_i \leq q$  if the  $i$ -th step is (Null or South-East) of height  $q$ , and  $1 \leq \xi_i \leq p + q + 1$  if the  $i$ -th step is (North or East) of abscissa  $p$  and height  $q$ .

Denote by  $\Delta_n^k$  the set of path diagrams of depth  $k$  and length  $n$ . The main ingredient of our proof is a mapping from  $\Delta_n^k$  to  $\mathcal{OP}_n^k$  which keeps track of several statistics of Steingrímsson.

**Theorem 2.5** There exists a bijection  $\psi : \Delta_n^k \rightarrow \mathcal{OP}_n^k$  such that if  $h = (\omega, \xi)$  is in  $\Delta_n^k$  and if the  $i$ -th step of  $w$  is of abscissa  $p$  and height  $q$ , and of type:

(i) North or East : then,  $i \in (\mathcal{O} \cup \mathcal{S})(\psi(h))$ ,  
 $(lcs + rcs)_i(\psi(h)) = q$  and  $(lsb + rsb)_i(\psi(h)) = p$ ,  
 $los_i(\psi(h)) = \xi_i - 1$  and  $ros_i(\psi(h)) = p + q + 1 - \xi_i$ .

(ii) South-East or Null : then,  $i \in (\mathcal{T} \cup \mathcal{C})(\psi(h))$ ,  
 $(lcs + rcs)_i(\psi(h)) = p$  and  $(lsb + rsb)_i(\psi(h)) = q - 1$ ,  
 $lsb_i(\psi(h)) = \xi_i - 1$  and  $rsb_i(\psi(h)) = q - \xi_i$

### 2.2 Generating functions of walks

For  $0 \leq k \leq n$ , let  $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$  and

$$Q_{n,k}(\mathbf{t}) := \sum_{\pi \in \mathcal{OP}_n^k} t_1^{(lcs+rcs)(\mathcal{O} \cup \mathcal{S})\pi} t_2^{(lcs+rcs)(\mathcal{T} \cup \mathcal{C})\pi} t_3^{rsb(\mathcal{T} \cup \mathcal{C})\pi} \times t_4^{lsb(\mathcal{T} \cup \mathcal{C})\pi} t_5^{ros(\mathcal{O} \cup \mathcal{S})\pi} t_6^{los(\mathcal{O} \cup \mathcal{S})\pi} t_7^{(lsb+rsb)(\mathcal{O} \cup \mathcal{S})\pi}. \tag{11}$$

Given a walk  $\omega$ , define the weight  $v(\omega)$  of  $\omega$  to be the product of the weights of all its steps, where the weight of a step of abscissa  $i$  and height  $j$  is:

$$\begin{cases} t_1^i t_7^j [i+j+1]_{t_5, t_6} & \text{if the step is North or East;} \\ t_2^i [j]_{t_3, t_4} & \text{if the step is Null or South-East.} \end{cases} \quad (12)$$

It follows easily from Theorem 2.5 that

$$\sum_{\omega \in \Omega_n^k} v(\omega) = Q_{n,k}(\mathbf{t}).$$

Denote by  $|\omega|$  the length of the walk  $\omega$ . Then, using the above identity, we get

$$Q_k(a; \mathbf{t}) := \sum_{n \geq 0} Q_{n,k}(\mathbf{t}) a^n = \sum_{w \in \Omega^k} v(w) a^{|\omega|}. \quad (13)$$

It is obvious that the number of vertices of  $D_k$  is equal to

$$\widehat{k} := 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Let  $v_1, \dots, v_{\widehat{k}}$  be the vertices of  $D_k$  arranged according to the following order:  $(i, j) \leq (i', j')$  if and only if  $i+j < i'+j'$  or  $(i+j = i'+j'$  and  $j \geq j')$ . For instance, we get  $v_1 = (0, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (1, 0)$ ,  $v_4 = (0, 2)$ ,  $v_5 = (1, 1)$ ,  $v_6 = (2, 0)$ ,  $\dots$ ,  $v_{\widehat{k}} = (k, 0)$ .

The adjacency matrix  $A_k$  of  $D_k$  relative to the valuation  $v$  is the  $\widehat{k} \times \widehat{k}$  matrix defined by

$$A_k(i, j) = \begin{cases} v(v_i, v_j) & \text{if } (v_i, v_j) \text{ is an edge of } D_k; \\ 0 & \text{otherwise.} \end{cases}$$

Applying the transfer-matrix method (see e.g. [9, Theorem 4.7.2]), we obtain

$$Q_k(a; \mathbf{t}) = \frac{(-1)^{1+\widehat{k}} \det(I - aA_k; \widehat{k}, 1)}{\det(I - aA_k)}, \quad (14)$$

where  $(B; i, j)$  denotes the matrix obtained by removing the  $i$ -th row and  $j$ -th column of  $B$  and  $I$  is the  $\widehat{k} \times \widehat{k}$  identity matrix.

In order to prove Theorem 1.9 it suffices to evaluate the following specializations of (14):

$$f_k(a; x, y, t, u) = Q_k(a; x, x, x, y, t, u, y), \quad (15)$$

$$g_k(a; z, t, u) = Q_k(a; 1, z, 1, z, t, u, 1). \quad (16)$$

Let  $A'_k$  and  $A''_k$  be the adjacency matrix of  $D_k$  relative to the weight function  $v'$  and  $v''$  obtained from the weight function  $v$  by making the substitution (15) and (16), respectively. Namely, the weights  $v'(e)$  and  $v''(e)$  of an edge  $e = ((i, j), (i', j'))$  of  $D_k$  with initial vertex  $(i, j)$  are :

$$v'(e) = \begin{cases} x^i y^j [i+j+1]_{t,u} & \text{if } (i', j') = (i, j+1) \text{ or } (i+1, j); \\ x^i [j]_{x,y} & \text{if } (i', j') = (i, j) \text{ or } (i+1, j-1), \end{cases}$$

and

$$v''(e) = \begin{cases} [i+j+1]_{t,u} & \text{if } (i', j') = (i, j+1) \text{ or } (i+1, j); \\ z^i [j]_z & \text{if } (i', j') = (i, j) \text{ or } (i+1, j-1). \end{cases}$$

Now, for each  $k \geq 0$  let

$$M_k = I - aA'_k \quad \text{and} \quad N_k = I - aA''_k.$$

Then by (14), (15) and (16) we have

$$f_k(a; x, y, t, u) = \frac{(-1)^{1+\widehat{k}} \det(M_k; \widehat{k}, 1)}{\det M_k}, \quad (17)$$

$$g_k(a; z, t, u) = \frac{(-1)^{1+\widehat{k}} \det(N_k; \widehat{k}, 1)}{\det N_k} \quad (18)$$

for each  $k \geq 1$ .

Since  $M_n$  and  $N_n$  are upper triangular matrices it is easy to see that for each  $n \geq 1$

$$\det M_n = \prod_{m=1}^n \prod_{i=0}^m (1 - ax^i[m-i]_{x,y}), \quad (19)$$

$$\det N_n = \prod_{m=1}^n \prod_{k=0}^{n-m} (1 - az^k[m]_q). \quad (20)$$

The evaluation of  $\det(M_n; \widehat{n}, 1)$  and  $\det(N_n; \widehat{n}, 1)$  is highly non trivial.

**Theorem 2.6** *Let  $n \geq 1$  be a positive integer. Then*

$$\det(M_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n x^{\binom{n}{2}} [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{i=1}^m (1 - ax^i[m-i+1]_{x,y}), \quad (21)$$

**Theorem 2.7** *Let  $n \geq 1$  be a positive integer. Then*

$$\det(N_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n [n]_{t,u}! \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} (1 - az^{k-1}[m]_z). \quad (22)$$

It is now trivial to obtain the following result.

**Corollary 2.8** *For  $k \geq 0$ , we have*

$$f_k(a; x, y, t, u) = \frac{a^k x^{\binom{k}{2}} [k]_{t,u}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \quad (23)$$

$$g_k(a; z, t, u) = \frac{a^k [k]_{t,u}!}{\prod_{i=1}^k (1 - az^{k-i}[i]_z)}. \quad (24)$$

Finally Theorem 1.9 follows immediately from Corollary 2.8 and the following lemma.

**Lemma 2.9** *The following identities hold:*

$$\phi_k(a; x, y, t, u) = f_k(a; x, y, xyt, uy^2), \quad (25)$$

$$\varphi_k(a; z, t, u) = g_k(az^{k-1}, 1/z, t, u/z). \quad (26)$$

Therefore in order to prove Theorem 1.9 it remains to prove Theorems 2.6 and 2.7.

### 3 Determinantal computations

#### 3.1 Proof of Theorem 2.6

The matrix  $M_n$  can be defined recursively by

$$M_0 = (1) \quad \text{and} \quad M_n = \left( \begin{array}{c|c} M_{n-1} & \overline{M}_{n-1} \\ \hline O_{n+1, \widehat{n-1}} & \widehat{M}_{n-1} \end{array} \right) \quad \text{for } n \geq 1, \quad (27)$$

where  $\widehat{M}_{n-1}$  is the  $(n+1) \times (n+1)$  matrix

$$\widehat{M}_{n-1} = (\delta_{ij} - ax^{i-1}[n+1-i]_{x,y}(\delta_{ij} + \delta_{i+1,j}))_{1 \leq i, j \leq n+1} \quad (28)$$

and  $\overline{M}_{n-1}$  is the  $\widehat{n-1} \times (n+1)$  matrix

$$\overline{M}_{n-1} = \left( \begin{array}{c} O_{\widehat{n-2}, n+1} \\ \check{M}_{n-1} \end{array} \right)$$



with the  $n \times (n + 1)$  matrix

$$\tilde{M}_{n-1} = (-ax^{i-1}y^{n-i}[n]_{t,u}(\delta_{ij} + \delta_{i+1,j}))_{1 \leq i \leq n, 1 \leq j \leq n+1}. \quad (29)$$

Here  $\delta_{ij}$  stands for the Kronecker delta and  $O_{m,n}$  denotes the  $m \times n$  zero matrix. Let

$$K_n = \widehat{n} - 1 = \frac{n(n+3)}{2},$$

and let  $P_n = (M_n; \widehat{n}, 1)$ , i.e the  $K_n \times K_n$  matrix obtained from  $M_n$  by deleting the  $\widehat{n}$ th row and the first column. The problem is to compute  $\det P_n$  for  $n \geq 1$ . The matrix  $P_n$  can also be defined recursively:

$$P_n = \left( \begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1} \end{array} \right)$$

Here  $\overline{P}_{n-1}$  is a  $K_{n-1} \times (n+1)$  matrix,  $X_{n-1}$  is a  $(n+1) \times K_{n-1}$  matrix, and  $\widehat{P}_{n-1}$  is a  $(n+1) \times (n+1)$  matrix. The idea here is to use the following well-known formula for any block matrix with an invertible square matrix  $A$ ,

$$\det \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det A \cdot \det (D - CA^{-1}B).$$

Since the entries of  $CA^{-1}B$  are also written by minors, we guess these entries and prove it by induction.

### 3.2 Proof of Theorem 2.7

Let  $F = \{F_n\}_{n=1}^{\infty}$  be a sequence of non-zero functions in finitely many variables  $v_1, v_2, \dots$ . We use the convention that  $F_n! = \prod_{k=1}^n F_k$  and

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_F = \begin{cases} \frac{F_n!}{F_k! F_{n-k}!}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

We prove Theorem 2.7 (22) by considering the following matrix  $N_n(x, a)$ , which generalize the matrix  $N_n$  (set  $x = 1$  and  $F_n = [n]_{t,u}$  to obtain  $N_n$ ). Let  $N_n(x, a)$  be the matrix defined inductively by:

$$N_0(x, a) = (x)$$

and

$$N_n(x, a) = \left( \begin{array}{c|c} N_{n-1}(x, a) & \overline{N}_{n-1}(x, a) \\ \hline O_{n+1, \widehat{n-1}} & \widehat{N}_{n-1}(x, a) \end{array} \right) \quad (30)$$

where  $\widehat{N}_{n-1}(x, a)$  is the  $(n+1) \times (n+1)$  matrix defined by

$$\widehat{N}_{n-1}(x, a) = (x\delta_{ij} - aq^{i-1}[n+1-i]_q(\delta_{ij} + \delta_{i+1,j}))_{1 \leq i, j \leq n+1} \quad (31)$$

and  $\overline{N}_{n-1}(x, a)$  is the  $\widehat{n-1} \times (n+1)$  matrix

$$\left( \begin{array}{c} O_{\widehat{n-2}, n+1} \\ \hline \tilde{N}_{n-1} \end{array} \right)$$

with the  $n \times (n+1)$  matrix

$$\tilde{N}_{n-1} = (-aF_n \cdot (\delta_{ij} + \delta_{i+1,j}))_{1 \leq i \leq n, 1 \leq j \leq n+1}. \quad (32)$$

Let  $\dot{N}_n(x, a)$  denote the matrix obtained from  $N_n(x, a)$  by deleting the  $\widehat{n}$ th row and the first column. Then the following theorem is sufficient to prove our result.

**Theorem 3.1** *We have*

$$\det \dot{N}_n(x, a) = (-1)^{\frac{n(n-1)}{2}} a^n F_n! x^n \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} (x - aq^{k-1}[m]_q). \quad (33)$$

Setting  $x = 1$  and  $F_n = [n]_{t,u}$  we obtain Theorem 2.7.

Here our strategy is as follows. We regard  $\det N_n(x, a)$  as a polynomial in  $x$  and find all linear factors. Finally we check the leading coefficient in the both sides.

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