Survival probability of a critical multi-type branching process in random environment

Elena Dyakonova

1Department of Discrete Mathematics, Steklov Mathematical Institute, Gubkin St. 8, 119991 Moscow, Russia

We study a multi-type branching process in i.i.d. random environment. Assuming that the associated random walk satisfies the Doney-Spitzer condition, we find the asymptotics of the survival probability at time \( n \) as \( n \to \infty \).

Keywords: branching processes in random environment, Doney-Spitzer condition, survival probability

Introduction

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, [1], [2], [4]-[7], [9]-[14]). A branching process in random environment was first generated by a sequence of independent identically distributed random variables under the condition \( E X^2 < \infty \) for the increment \( X \) of the associated random walk was found in [6], [9] for single-type processes, and in [4] for multi-type processes. Recent papers [1], [5], [12]-[14] study the survival probability for an extended class of the critical single-type branching processes in random environment. This process can be described as follows.

Let \( Z(n) = (Z_1(n), ..., Z_p(n)) \), \( n = 0, 1, ... \), be a \( p \)-type Galton-Watson branching process in a random environment. This process can be described as follows.

Let \( N_0 = \{0, 1, 2, ... \} \) and \( N_0^p \) be the set of all vectors \( t = (t_1, ..., t_p) \) with non-negative integer coordinates. Denote by \( (\Delta_1, B(\Delta_1)) \) a set of probability measures on \( N_0^p \) with \( \sigma \)-algebra \( B(\Delta_1) \) of Borel sets endowed with the metric of total variation, and by \( (\Delta, B(\Delta)) \) the \( p \)-times product of the space \( (\Delta_1, B(\Delta_1)) \) on itself. Let \( F = (F^{(1)}, ..., F^{(p)}) \) be a random variable (random measure) taking values in \( (\Delta, B(\Delta)) \). An infinite sequence \( \Pi = (F_0, F_1, F_2, ...) \) of independent identically distributed copies of \( F \) is said to form a random environment and we will say that \( F \) generates \( \Pi \). A sequence of random \( p \)-dimensional vectors \( Z(0), Z(1), Z(2), ... \) with non-negative integer coordinates is called a \( p \)-type branching process in random environment \( \Pi \), if \( Z(0) \) is independent of \( \Pi \) and for all \( n \geq 0, z = (z_1, ..., z_p) \in N_0^p \) and \( f_0, f_1, ..., \in \Delta \)

\[
\mathcal{L}(Z(n+1) \mid Z(n) = (z_1, ..., z_p), \Pi = (f_1, f_2, ...)) \\
= \mathcal{L}(Z(n+1) \mid Z(n) = (z_1, ..., z_p), F_n = f_n) \\
= \mathcal{L}(\varepsilon^{(1)}_{n,1} + \cdots + \varepsilon^{(1)}_{n,z_1}, \varepsilon^{(2)}_{n,1} + \cdots + \varepsilon^{(2)}_{n,z_2}, \cdots, \varepsilon^{(p)}_{n,1} + \cdots + \varepsilon^{(p)}_{n,z_p}), \tag{1}
\]

where \( f_n = (f^{(1)}_n, f^{(2)}_n, ..., f^{(p)}_n) \in \Delta, \varepsilon^{(i)}_{n,1}, \varepsilon^{(i)}_{n,2}, ..., \varepsilon^{(i)}_{n,z_i}, \ v = 1, ..., p, \) are independent \( p \)-dimensional random vectors, and for each \( i = 1, ..., p \) the random vectors \( \varepsilon^{(i)}_{n,1}, \varepsilon^{(i)}_{n,2}, ..., \varepsilon^{(i)}_{n,z_i} \) are identically distributed according to the measure \( f^{(i)}_n \). Relation (1) defines a branching Galton-Watson process \( Z(n) \) in random...
environment which describes the evolution of a particle population \( Z(n) = (Z_1(n), ..., Z_p(n)), ~ n = 0, 1, ..., \) where \( Z_i(n), \ i = 1, ..., p, \) is the number of type \( i \) particles in the \( n \)-th generation.

This population evolves as follows. If \( F_n = f_n \) then each of the \( Z_i(n) \) particles of type \( i \) existing at the time \( n \), produces offspring in accordance with the \( p \)-dimensional probability measure \( f^{(i)}_n \) independently of the reproduction of other particles. Thus, the \( i \)-th component of the vector \( Z(n + 1) = (Z_1(n + 1), ..., Z_p(n + 1)) \) is equal to the number of type \( i \) particles among all direct descendants of the particles of the \( n \)-th generation. The distribution of \( Z(0) \) will be specified later.

The main results

Let \( J^p \) be the set of all column vectors \( s = (s_1, ..., s_p)^T, \ 0 \leq s_i \leq 1, i = 1, ..., p. \) For \( s \in J^p \) and \( t \in \mathbb{N}_0 \) set \( s^t = \prod_{i=1}^{p} s_i^t. \) Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with \( F = (F^{(1)}, ..., F^{(p)}) \) generating \( \Pi \) a random \( p \)-dimensional column vector \( F(s) = (F^{(1)}(s), ..., F^{(p)}(s))^T, \ s \in J^p, \) whose components are \( p \)-dimensional (random) generating functions \( F^{(i)}(s) \) corresponding to \( F^{(i)}, \ 1 \leq i \leq p: \)

\[
F^{(i)}(s) = \sum_{t \in \mathbb{N}_0^p} F^{(i)}(\{t\}) s^t, \ s \in J^p.
\]

In a similar way we associate with the component \( F_n = (F^{(1)}_n, ..., F^{(p)}_n), \ n \geq 0, \) of the random environment
\( \Pi = (F_0, F_1, F_2, ...) \) a random vector \( F_n(s) = (F^{(1)}_n(s), ..., F^{(p)}_n(s))^T, \ s \in J^p, \) the components of which are multidimensional (random) generating functions \( F^{(i)}_n(s) \), corresponding to \( F^{(i)}_n, \ 1 \leq i \leq p, \)

\[
F^{(i)}_n(s) = \sum_{t \in \mathbb{N}_0^p} F^{(i)}_n(\{t\}) s^t.
\]

Let \( e_j, j = 1, ..., p, \) be the \( p \)-dimensional row vector whose \( j \)-th component is equal to 1 and the others are zeros, \( \overline{0} = (0, ..., 0) \) be the \( p \)-dimensional row vector all whose components are zeros, and let \( \overline{1} = (1, ..., 1)^T \) be the \( p \)-dimensional column vector all whose components are equal to 1. For \( x = (x_1, ..., x_p) \) and \( y = (y_1, ..., y_p)^T \) we set \( |x| = \sum_{i=1}^{p} |x_i|, \ |y| = \sum_{i=1}^{p} |y_i|, \ (x, y) = \sum_{i=1}^{p} x_i y_i. \)

Let \( A = ||A(i,j)||_{i,j=1}^p \) be an arbitrary positive \( p \times p \) matrix. Denote by \( \rho(A) \) the Perron root of \( A \) and by \( u(A) = (u_1(A), ..., u_p(A))^T \) and \( v(A) = (v_1(A), ..., v_p(A)) \) the right and left eigenvectors of \( A \) corresponding to the eigenvalue \( \rho(A) \) and such that

\[
|v(A)| = 1, \ (v(A), u(A)) = 1.
\]

For vector-valued generating functions \( F(s) \) and \( F_n(s) \) we introduce the mean matrices

\[
M = M(F) = ||M(i,j)||_{i,j=1}^p = \left\| \frac{\partial F^{(i)}(\overline{1})}{\partial s_j} \right\|_{i,j=1}^p
\]

and

\[
M_n = M_n(F_n) = ||M_n(i,j)||_{i,j=1}^p = \left\| \frac{\partial F^{(i)}_n(\overline{1})}{\partial s_j} \right\|_{i,j=1}^p.
\]

Let \( C_\alpha, 0 < \alpha < 1, \) be the class of all matrices \( A = ||A(i,j)||_{i,j=1}^p \) such that

\[
\alpha \leq \frac{A(i_1,j_1)}{A(i_2,j_2)} \leq \alpha^{-1}, \ 1 \leq i_1, i_2, j_1, j_2 \leq p.
\]

One of our basic hypotheses is the following condition.

**Assumption A0.** There exist a number \( 0 < \alpha < 1 \) and a positive row vector \( v = (v_1, ..., v_p), \ |v| = 1, \) such that, with probability 1

\[
M = M(F) \in C_\alpha,
\]

and

\[
vM = \rho(M)v.
\]

(2)
Set \( p = \rho(M), \rho_n = \rho(M_n), n \geq 0 \). It is not difficult to see that in our settings \( X := \ln p, X_i := \ln \rho_{i-1}, i \geq 1 \), are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk \( S = (S_0, S_1, \ldots) \), where

\[ S_n = X_1 + \cdots + X_n, \ n \geq 1, \ S_0 = 0. \]

**Assumption A1.** There exists a number \( 0 < a < 1 \) such that

\[ \mathbb{P}(S_n > 0) \to a, \ n \to \infty. \]  

(3)

Extending the known classification of single-type branching processes in random environment (see [1], [12]), we call a \( p \)-type branching process \( Z(n), n \geq 0 \), in random environment II critical if its associated random walk is of the oscillating type, i.e., \( \limsup_{n \to \infty} S_n = +\infty \) a.s. and \( \liminf_{n \to \infty} S_n = -\infty \) a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical \( p \)-type branching processes in random environment.

Let \( 0 =: \gamma_0 < \gamma_1 < \ldots \) be the strict descending ladder epochs of \( S \). Put

\[ V(x) := \sum_{i=0}^{\infty} \mathbb{P}(S_{\gamma_i} \geq -x), \ x \geq 0; \ V(x) = 0, \ x < 0. \]

Since \( S \) is oscillating, the following relation holds [3]:

\[ \mathbb{E}V(x + X) = V(x), \ x \geq 0. \]  

(4)

For \( d \in \mathbb{N} \) set

\[ O_d = \{ t = (t_1, \ldots, t_p) \in \mathbb{N}_0^p \mid t_i < d, i = 1, \ldots, p \}, \ U_d = \mathbb{N}_0^p \setminus O_d. \]

Introduce the random variable

\[ \kappa(d) = \sum_{t \in U_d} \sum_{i=1}^{p} v_i \sum_{j,k=1}^{\mathbb{N}} \mathbb{P}^{(i)}\{t_j,t_k\}/\rho^2, \ d \in \mathbb{N}_0, \]

where \( v = (v_1, \ldots, v_p) \) is from [2]. Our next condition is connected with the random variable \( \kappa(d) \), which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

**Assumption A2.** There exist \( \varepsilon > 0 \) and \( d \in \mathbb{N}_0 \) such that

\[ \mathbb{E}(\ln^+ \kappa(d))^{1/a + \varepsilon} < \infty, \ \mathbb{E}\left(V(X)(\ln^+ \kappa(d))^{1/a + \varepsilon}\right) < \infty. \]

Let \( T = \min\{n \geq 0 : Z(n) = 0\} \) be the extinction moment for \( Z(n) \). Introduce the random variables

\[ Q^{(i)}(n) = \mathbb{P}(T > n|Z(0) = e_i, \Pi), \ Q(n) = (Q^{(1)}(n), \ldots, Q^{(p)}(n)), \]

and let

\[ q_i(k) = \mathbb{P}(T > k|Z(0) = e_i) = \mathbb{E}Q^{(i)}(k). \]

Note that under Assumptions A0 and A1 \( Q^{(i)}(n) \to 0 \) \( \mathbb{P} \)-a.s. as \( n \to \infty \) for all \( 1 \leq i \leq p \), since \( \mathbb{P} \)-a.s.

\[ (v, Q(n)) \leq \min_{0 \leq k \leq n-1} |vM_0 \cdots M_k| \leq \exp\{\min_{0 \leq k \leq n-1} S_k\} \to 0 \]

as \( n \to \infty \). Denote by \( u(n) = (u_1(n), \ldots, u_p(n))^T := u(M_0 \cdots M_n) \), \( n \geq 0 \), the right eigenvector of the product \( M_0 \cdots M_n \), corresponding to the Perron root \( \rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n \). To investigate the asymptotic behavior of \( q_i(n) \) and \( Q^{(i)}(n) \) as \( n \to \infty \) we need the following statement describing the behavior of \( u(n) \).

**Theorem 1** If Assumption A0 is valid, then there exist a random vector \( u = (u_1, \ldots, u_p)^T \) and a function \( g(n) \geq 0, g(n) \to 0, n \to \infty \), such that with probability 1

\[ |u_i(n) - u_i| \leq g(n), \ i = 1, \ldots, p. \]

In addition,

\[ (v, u) = 1, \ \alpha \leq u_i \leq 1/\alpha^*, \]

where \( \alpha^* = \min(v_1, \ldots, v_p) \) and \( v = (v_1, \ldots, v_p) \) is from [2].
The following statement describes the behavior of \( Q(n) \) as \( n \to \infty \).

**Theorem 2** Assume Assumptions A0 and A1. Then \( P \)-a.s., as \( n \to \infty \),

\[
\frac{Q_i(n)}{(v; Q(n))} \to u_i, \ i = 1, \ldots, p,
\]

where \( u = (u_1, \ldots, u_p) \) is from Theorem [1].

Now we are ready to formulate the main result of the paper.

**Theorem 3** Assume Assumptions A0, A1, and A2. Then, as \( n \to \infty \),

\[
q_i(n) \sim c_i n^{-(1-s)} l(n), \ c_i > 0, \ i = 1, \ldots, p,
\]

where \( l(n) \) is a function slowly varying at infinity.

Note that under our approach one of the key facts to prove Theorems [1][2][3] is convergence in distribution, as \( n \to \infty \), of the products \( \prod_{i=0}^{n-1} M_{i} \rho_{i}^{-1} \) of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known [8] that for \( p = 2 \) the products \( \prod_{i=0}^{n-1} M_{i} \rho_{i}^{-1} \) of the positive bounded independent identically distributed \( 2 \times 2 \) matrices \( A_i = M_{i} \rho_{i}^{-1} \) converges in distribution, as \( n \to \infty \), to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices \( A_i \) have a common positive right or left eigenvector. Hence, for the 2-type process \( Z(n) \) our assumption on existence of a common positive left eigenvector of the matrices \( M \) is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of their increments, as well as all non-degenerate symmetric random walks. In these cases \( a = 1/2 \).

Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid and Assumption A1' is fulfilled).

Using the proof of Theorem 3 and results of paper [1], one can obtain also the following statement.

**Theorem 4** Assume Assumptions A0, A1', and A2'. Then the statement of Theorem 3 remains true.
References


